Errors in Dielectric Measurements Due to a Sample Insertion Hole in a Cavity*

A. J. ESTIN†, MEMBER, IRE, AND H. E. BUSSEY†, SENIOR MEMBER, IRE

Summary—**The measurement of complex permittivities of isotropic media at microwave frequencies is performed with high precision by means of cylindrical cavity resonators. However, a hole in the cavity wall, for inserting the sample causes a frequency pulling of the resonator, which in turn introduces an error in the measured dielectric constant. These effects are measured, and with perturbation theory as a guide, correction factors are developed.**

Introduction

THE cavity resonator has assumed a prominent role in the microwave determination of dielectric parameters of materials.[1,2] Among the more useful of the resonant modes, particularly at 10 cm and longer wavelengths, are the cylindrical TM_{0m0} family. One sample geometry conformable to this mode is a concentric post equal in height to that of the cavity. In order to introduce such a sample into the cavity, without removing the lid each time, it is convenient to have a centered hole in one endplate. If the sample is made somewhat longer than the height of the cavity, then this hole will in addition provide concentric alignment. The redistribution of fields due to this hole, however, results in a frequency-pulling, which in turn produces an error in the measured value of the dielectric constant of the material of the post.

The hole causes a departure from the geometrically simple shape of an idealized cavity, and the exact solution becomes exceedingly difficult. Accordingly, a solution is usually attempted in terms of some approximate perturbation or variational method.[3,4] With certain of these mathematical techniques, the limits of error or order of magnitude of error can be predicted.[4] Nevertheless, even with these, and particularly with perturbation methods, an experimental check of the predicted results is of vital importance. This paper provides experimental data of a high quality suitable both for direct empirical use and for experimental verification of any theoretical approach to the problem. Furthermore, a perturbation development is presented in this paper, and is intended as a general guide to show the expected functional dependence.

Instrumentation

A cavity was fabricated which resonated in the TM_{010} mode at 509 Mc and reresonated in the TM_{020} mode at 1168 Mc. This cavity (Fig. 1) was made with an axial hole in one endplate, which in turn opened outward into a tube of the same diameter. Construction was such as to permit either the insertion of various sizes of metal tubular sample guides, or the use of a solid plug to enclose the sample for closely approximating the closed ("perfect") cavity. Samples of various materials were carefully machined to a right cylindrical shape to fit under the solid plug with a clearance of no more than 0.002 inch. Additional cylinders were made from the same materials to permit extension of the dielectric post out through the hole and tube in the cases where the hole pulling was to be examined. Three sizes of tubes were employed in order to examine the dependence of error on hole size. In this way a direct comparison was made between the true resonant frequencies and di-

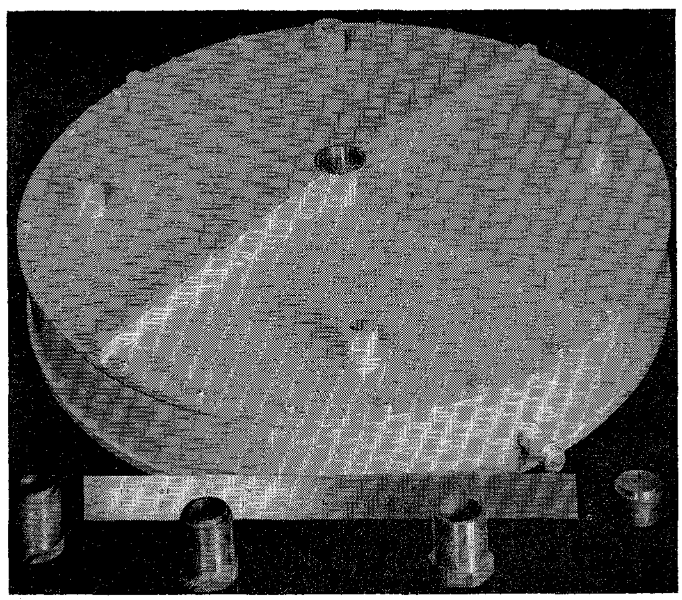

Fig. 1—TM_{0m0}-mode cavity with axial hole. The tubular sample guides and the plug are shown. The diameter is 17.8 inches and the height is 2.25 inches.

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† National Bureau of Standards, Boulder, Colo.

[1] G. Birnbaum and J. Franeau, "Measurement of the dielectric constant and loss of solids and liquids by a cavity perturbation method," *J. Appl. Phys.*, vol. 20, pp. 817–818; August, 1949.

[2] F. Horner, *et al.*, "Resonance methods of dielectric measurement at centimeter wavelengths," *J. IEE*, vol. 93, pt. III, pp. 55–57; January, 1946.

[3] H. A. Bethe and J. Schwinger, "Perturbation Theory of Resonant Cavities," NDRC, Rept. No. D1-117; March, 1943. Their development is for a hole in a thin wall; the problem considered in this paper is where the cavity wall is so thick that the hole is in fact an "infinitely long" tube.

[4] P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2; 1953. (See especially ch. 9.)

electric constant as measured with the plug closing the cavity, and the pulled resonant frequencies and apparent dielectric constant obtained with a hole and long tube extension.

Possible sources of experimental inaccuracy were thoroughly investigated to determine the limitations of the investigation. It was found that the effects of iris pulling, of imperfect closing of the plug, and of cavity and sample asymmetries were negligible. The most serious systematic error present was the tolerance of 0 to −0.002 inch on the length of the post. However, within the tolerances held, this error could not affect the hole pulling results by more than 5 per cent at a dielectric constant of eight, and the effect rapidly diminished for lower values. The remaining limitations, such as imperfect frequency resolution of the cavity and inexact frequency measurements were random and are indicated by the dispersion of the results.

Under the assumption that the Q of the cavity is high and that the loss tangent of the dielectric material is small, the real part of the dielectric constant is readily determined from a calculation involving only the frequency shift, the physical dimensions of the system, and a knowledge of the mode of oscillation involved. Exact solution of the boundary value problem yields the following for the real part of the dielectric constant,[2]

$$\frac{\sqrt{\kappa}\, ka J_1(\sqrt{\kappa}\, ka)}{J_0(\sqrt{\kappa}\, ka)} = \frac{J_1(ka) - \dfrac{J_0(kb)}{Y_0(kb)} Y_1(ka)}{J_0(ka) - \dfrac{J_0(kb)}{Y_0(kb)} Y_0(ka)}\, ka, \quad (1)$$

where a is the sample radius, b is the cavity radius, k is the wave number of the cavity resonated with sample, κ is the real part of the relative dielectric constant, and J_n and Y_n are Bessel functions of order n of the first and second kinds. A numerical solution of this equation (along with other relations for the imaginary part of the dielectric constant) has been programmed for high-speed digital computation. All experimental results in this paper were calculated by this method.

The measurement at 1100 Mc was made by means of an improved cavity Q-meter technique.[5] However, the lack of a conveniently swept oscillator source at 500 Mc necessitated a different technique. In brief, a stable, continuously tunable, CW signal generator provided the signal; a portion of this signal was mixed with an arbitrary fixed frequency (usually 500 Mc) derived from the National Bureau of Standards' 100-kc primary standard, and the difference was then directly measured to an accuracy of ±1 kc.

[5] C. G. Montgomery, "Technique of Microwave Measurements," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, sec. 6.29; 1948.

Theory

The theoretical evaluation of the effect of the hole upon the dielectric constant necessitates two steps. First, an expression is required for the frequency pulling of a cavity by a dielectric-filled hole. Second, this expression must be substituted into one for the error in dielectric constant as a function of errors in resonant frequencies.

The frequency pulling of the hole may be calculated by applying the Adiabatic Invariance Theorem (Boltzmann-Ehrenfest Principle).[6] This states that for an oscillating electromagnetic system, the relative frequency pulling is equal to the relative energy change of the system due to changing its configuration. One thinks of the hole as being created by pulling a disk section of the cavity wall the size of the hole to infinity against the electromagnetic stress on the surface, $(\epsilon E^2 - \mu H^2)/2$, while the dielectric column follows, filling the hole like a liquid. The energy change is obtained from the work performed. It will now be assumed, as in perturbation theory, that the frequency changes as a result of inserting the dielectric post and creating the hole are small compared with the empty resonant frequency of the cavity. This is insured by requiring that

$$b^2/m^2 \gg a^2\kappa \quad (2)$$

where m is the radial index of the TM_{0m0} mode of the cavity and a is the radius of the hole, identical to the sample radius. Eq. (2) insures that the wave in the dielectric-filled tube is evanescent, and that the dielectric perturbation is small.

In order to calculate the work of removing the disk to infinity, it is necessary to estimate the fields against which this work is done. The circularly symmetric TM mode in the cavity insures that only TM_{0j} modes will be excited in the tube. We shall approximate the electric field in the tube by the TM_{01} mode only:

$$E = AJ_0\left(\frac{\rho_{01} r}{a}\right) e^{-j\gamma z}, \quad (3)$$

where A is the amplitude factor, z is the distance into the tube as measured outward from the interface between the cavity and the tube, and γ is the propagation constant in the tube of the TM_{01} mode.

For conditions where the tube is far below cutoff, $\gamma \approx -j\rho_{01}/a$. We rather arbitrarily equate the integrated square of the unperturbed cavity field at the interface to the integrated square of the tube field at the interface. Thus the amplitude factor in (3) is given by

$$A = E_0/J_1(\rho_{01}) \quad (4)$$

where E_0 is the amplitude factor of the electric field in the cavity.

[6] W. R. MacLean, "The resonator action theorem," *Quart. Appl. Math.*, vol. 2, pp. 329–335; January, 1945. See also J. Bernier, "Sur les cavites electromagnetiques, *L'Onde Elect.*, vol. 26, pp. 305–317; August–September, 1946.

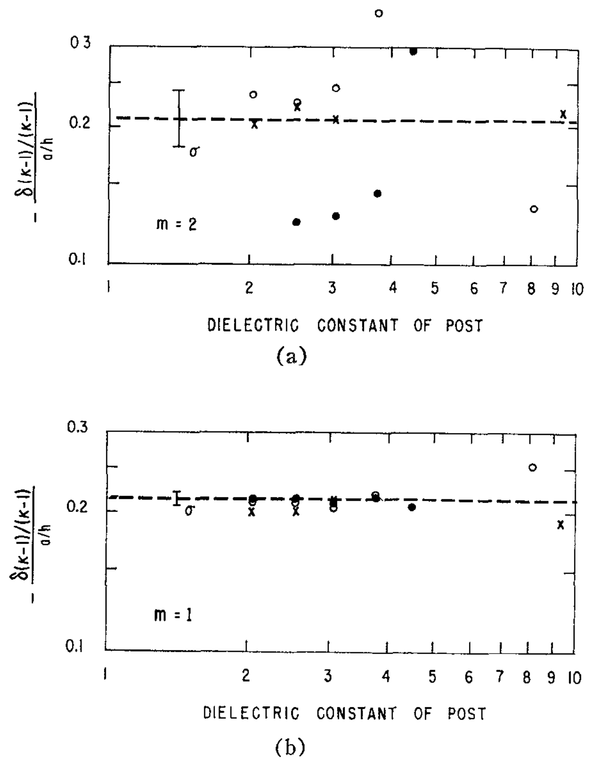

Fig. 2—Experimental points showing that

$$\frac{\delta(\kappa-1)/(\kappa-1)}{a/h}$$

is nearly constant. (a) TM_{010} mode at 509 Mc, (b) TM_{020} mode at 1168 Mc.
● $a=0.3125$ inch
× $a=0.375$ inch
○ $a=0.500$ inch.

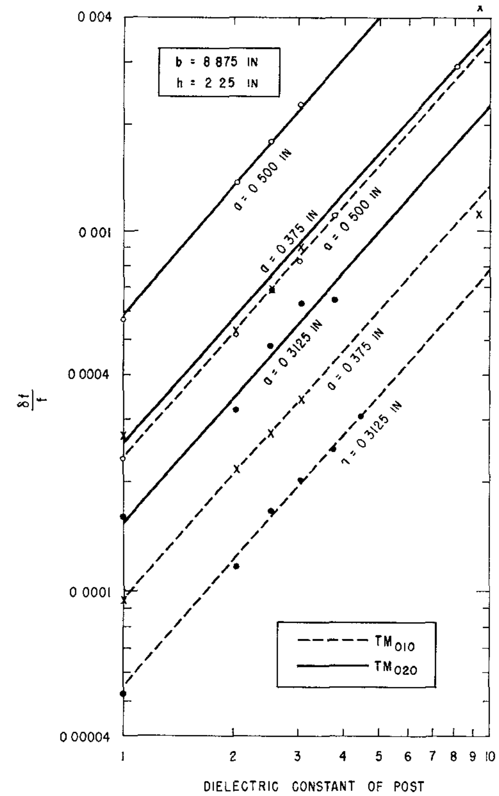

Fig. 3—Experimental frequency shift due to an axial hole of radius a, with various dielectric materials in the cavity and extending out through the hole.

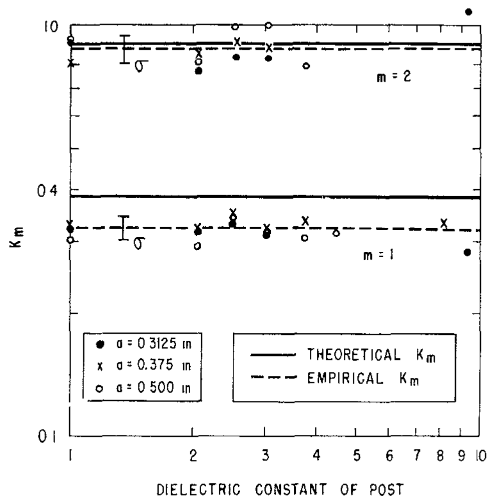

Fig. 4—The fit of the experimental data to the constant K_m defined by (8).

Combining (3) and (4) into the Adiabatic Invariance Theorem, we arrive at an expression for the frequency pulling:

$$\frac{\delta f}{f} = \frac{a^3\kappa}{4b^2 h \rho_{01} J_1^2(\rho_{0m})} \tag{5}$$

where h is the height of the cavity and ρ_{0m} is the mth root of J_0, corresponding to the TM_{0m0} mode in the cavity, and where h is 2.25 inches.

The first-order perturbation relation for dielectric constant in terms of frequency shift[1,3] is

$$\kappa - 1 = 2J_1^2(\rho_{0m}) \frac{b^2}{a^2}\left(\frac{f_e}{f_s} - 1\right), \tag{6}$$

where f_s is the resonant frequency with the sample inserted, and f_e is the resonant frequency of the empty cavity. The error in $(\kappa-1)$ due to errors in f_s and f_e is easily obtained from (6), and with the expression for the error in frequency, (5), this error in dielectric constant is finally:

$$\frac{\delta(\kappa - 1)}{\kappa - 1} = -\frac{a}{2\rho_{01}h}\frac{f_e}{f_s} \approx -\frac{1}{2\rho_{01}}\frac{a}{h}. \tag{7}$$

Results and Discussion

A series of direct measurements was made on materials of relative dielectric constants between 1 and 10. Ratios of a/h up to 0.22 were investigated. In no case did the value of $m^2a^2\kappa/b^2$ exceed 0.11.

If the experimental values of

$$\frac{\delta(\kappa - 1)}{(\kappa - 1)} \Big/ \frac{a}{h}$$

are plotted as a function of κ (Fig. 2), (7) predicts that the curve will be a constant function with value equal to $1/2\rho_{01}=0.208$, independent of κ and of the radial index of the mode of oscillation within the TM_{0m0} family. Experimentally, this value was found to be 0.21 for the TM_{010} mode and 0.2 for the TM_{020} mode.

The experimental behavior of the normalized frequency shift was next investigated. If a log-log plot is made of $\delta f/f$ against κ, as in Fig. 3, a dependence on some power of κ higher than unity is indicated, *i.e.*, $\kappa^{1+\Delta}$. From the slope, Δ was found to be about 0.16. The approximate theory developed herein accounts only for $\Delta=0$. However, a strict first-order perturbation theory[7] indicates that Δ is greater than zero.

Therefore, we shall empirically modify (5) to

$$\frac{\delta f}{f} = K_m \frac{a^3\kappa^{1+\Delta}}{b^2 h}, \tag{8}$$

where $\Delta=0.16$ and K_m is a function only of m, the radial mode index. The theory (5) gives $K_1=0.386$ and $K_2=0.897$. Experimentally the values of K_1 and K_2 are found to be 0.32 and 0.87, respectively. Fig. 4 shows the experimental data that determined K_m.

It is interesting to note that although the simple equation (5) departs significantly from the experimental equation (8), this causes no discernible error in (7) which depends on (5). [It may be seen that (7) is well supported by the experimental data in Fig. 2.] This has not been investigated in detail, but it would seem that (6) also departs from experiment, and that when (5) and (6) are combined to obtain (7) there is a cancelling of errors.

Acknowledgment

The authors are indebted to Dr. George Birnbaum for his early leadership in problems of cavity dielectric measurements, out of which this investigation arose.

[7] D. M. Kerns and H. E. Bussey, manuscript in preparation.

Correction

D. S. Lerner and H. A. Wheeler, authors of "Measurement of Bandwidth of Microwave Resonator by Phase Shift of Signal Modulation," which appeared on pages 343–345 of the May, 1960, issue of these Transactions, have brought the following to the attention of the *Editor*.

Reference [8], which appears on page 345, should read:

F. H. James, "A method for the measurement of high Q-factors," *Proc. IEE*, vol. 106, pt. B, pp. 489–492; September, 1959. (Recent proposal of the subject method.)